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1997 J. Phys. A: Math. Gen. 30 5405

(http://iopscience.iop.org/0305-4470/30/15/025)

J. Phys. A: Math. Gen. **30** (1997) 5405–5409. Printed in the UK PII: S0305-4470(97)83547-0

The reversed q-exponential functional relation

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Received 21 April 1997

Abstract. After obtaining some useful identities, we prove an additional functional relation for q-exponentials with reversed order of multiplication, as well as the well known direct one, in a completely rigorous manner.

1. Introduction

One of the most appealing results to come out of q-analysis is that the q-exponential function, defined by ${}_qD_x \exp_q(x) = \exp_q(x)$, where ${}_qD_x$ is the q-derivative, also satisfies the same defining functional relationship for ordinary exponential functions (up to normalization), given by

$$F(x)F(y) = F(x+y) \tag{1}$$

provided that $xy = q^{-1}yx$ (that is, (x, y) belongs to the Manin quantum plane). This result was first found by Schützenberger [1] long before the non-commutative aspects of q-analysis were generally recognized and has been rediscovered many times subsequently, for example in [2, 3]. It can be proved by means of q-combinatorics [1, 2], or by an argument based on the definition of the q-exponential as an eigenfunction of the q-derivative [3].

Besides the above well known result, there is, in fact, an additional functional relation in the opposite order for the q-exponential functions, which is not so well known given by

$$F(y)F(x) = F(x+y+(1-q^{-1})yx) \tag{2}$$

provided that the same condition $xy = q^{-1}yx$ holds. We first became aware of this relationship in the work of Faddeev and Yu Volkov in their study of lattice Virasoro algebra [4], when they obtained a similar result in the case of a different realization of the q-exponential, in terms of an infinite product. Their definition of the q-exponential suffered from the drawback that it did not go over into the ordinary exponential function in the commuting limit $q \to 1$. In this paper, we will provide a completely rigorous proof of the reverse functional relation in the form stated in (2). The proof is somewhat tricky in that a seemingly unrelated identity has to be obtained first as an intermediate step.

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0305-4470/97/155405+05$19.50

2. Proof of the reversed q-exponential functional relation

For completeness we quickly review Schützenberger and Cigler's result, which will be used in our subsequent proof:

$$\exp_q x \exp_q y = \exp_q(x+y) \qquad \text{if } xy = q^{-1}yx \tag{3}$$

where

$$\exp_q x \equiv \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \qquad [n] \equiv \sum_{k=0}^{n-1} q^k \qquad [n]! \equiv [n][n-1]\cdots[1].$$

Proof.

$$\begin{aligned}
\exp_q x \exp_q y &= \left(\sum_{m=0}^{\infty} \frac{x^m}{[m]!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{[n]!}\right) \\
&= \sum_{k=0}^{\infty}\sum_{r=0}^{k} \frac{x^r y^{k-r}}{[r]![k-r]!} \\
&= \sum_{k=0}^{\infty} \frac{1}{[k]!}\left(\sum_{r=0}^{k} \frac{[k]!}{[r]![k-r]!} x^r y^{k-r}\right) \\
&= \sum_{k=0}^{\infty} \frac{(x+y)^k}{[k]!} \qquad \text{(by (A1), see the appendix)} \\
&= \exp_q(x+y).
\end{aligned}$$

□

Now let us go on to prove the following formula:

$$\begin{aligned}
x^n &= 1 + \sum_{r=1}^{n} \frac{[(q^{n-r+1}-1)(q^{n-r+2}-1)\cdots(q^n-1)][(x-1)(x-q)\cdots(x-q^{r-1})]}{(q-1)(q^2-1)\cdots(q^r-1)} \\
&\equiv \sum_{r=0}^{n} \frac{[(q^{n-r+1}-1)(q^{n-r+2}-1)\cdots(q^n-1)][(x-1)(x-q)\cdots(x-q^{r-1})]}{(q-1)(q^2-1)\cdots(q^r-1)}.
\end{aligned} \tag{4}$$

Proof. Suppose for some $n=k$ we have

$$x^k = \sum_{r=0}^{k} \frac{[(q^{k-r+1}-1)(q^{k-r+2}-1)\cdots(q^k-1)][(x-1)(x-q)\cdots(x-q^{r-1})]}{(q-1)(q^2-1)\cdots(q^r-1)}.$$

Now, consider x^{k+1},

$$\begin{aligned}
x^{k+1} = &\sum_{r=0}^{k} \frac{[(q^{k-r+1}-1)(q^{k-r+2}-1)\cdots(q^k-1)][(x-1)(x-q)\cdots(x-q^r)]}{(q-1)(q^2-1)\cdots(q^r-1)} \\
&+ \sum_{r=0}^{k} \frac{q^r[(q^{k-r+1}-1)(q^{k-r+2}-1)\cdots(q^k-1)][(x-1)(x-q)\cdots(x-q^{r-1})]}{(q-1)(q^2-1)\cdots(q^r-1)} \\
= &(x-1)(x-q)\cdots(x-q^k) \\
&+ \sum_{r=0}^{k-1} \frac{[(q^{k-r+1}-1)(q^{k-r+2}-1)\cdots(q^k-1)][(x-1)(x-q)\cdots(x-q^r)]}{(q-1)(q^2-1)\cdots(q^r-1)}
\end{aligned}$$

$$+\sum_{r=0}^{k}\frac{q^r[(q^{k-r+1}-1)(q^{k-r+2}-1)\cdots(q^k-1)][(x-1)(x-q)\cdots(x-q^{r-1})]}{(q-1)(q^2-1)\cdots(q^r-1)}$$

$$=(x-1)(x-q)\cdots(x-q^k)$$

$$+\sum_{r=1}^{k}\frac{[(q^{k-r+2}-1)(q^{k-r+3}-1)\cdots(q^k-1)][(x-1)(x-q)\cdots(x-q^{r-1})]}{(q-1)(q^2-1)\cdots(q^{r-1}-1)}$$

$$+\sum_{r=0}^{k}\frac{q^r[(q^{k-r+1}-1)(q^{k-r+2}-1)\cdots(q^k-1)][(x-1)(x-q)\cdots(x-q^{r-1})]}{(q-1)(q^2-1)\cdots(q^r-1)}$$

$$=\sum_{r=0}^{k+1}\frac{[(q^{(k+1)-r+1}-1)(q^{(k+1)-r+2}-1)\cdots(q^{k+1}-1)][(x-1)(x-q)\cdots(x-q^{r-1})]}{(q-1)(q^2-1)\cdots(q^r-1)}.$$

Since, for $n=1$, obviously we have

$$x=\sum_{r=0}^{1}\frac{[(q^{1-r+1}-1)(q^{1-r+2}-1)\cdots(q^1-1)][(x-1)(x-q)\cdots(x-q^{r-1})]}{(q-1)(q^2-1)\cdots(q^r-1)}$$

the proof is complete. □

There follows another identity which is a simple consequence of the previous one:

$$\sum_{r=0}^{m\text{ or }n}\frac{q^{r(r-1)/2-mn}(q-1)^r}{[m-r]![n-r]![r]!}=\frac{1}{[m]![n]!}. \tag{5}$$

Proof.

$$\sum_{r=0}^{n}\frac{q^{r(r-1)/2-mn}(q-1)^r}{[m-r]![n-r]![r]!}$$

$$=\sum_{r=0}^{n}\{q^{r(r-1)/2-mn}(q-1)^r([m-r+1][m-r+2]\cdots[m])$$

$$\times([n-r+1][n-r+2]\cdots[n])\}\{[m]![n]![r]!\}^{-1}$$

$$=\sum_{r=0}^{n}\{q^{r(r-1)/2-mn}[(q^{m-r+1}-1)(q^{m-r+2}-1)\cdots(q^m-1)]$$

$$\times[(q^{n-r+1}-1)(q^{n-r+2}-1)\cdots(q^n-1)]\}$$

$$\times\{[m]![n]!(q-1)(q^2-1)\cdots(q^r-1)\}^{-1}$$

$$=\frac{1}{[m]![n]!}\sum_{r=0}^{n}\{[(q^m-1)(q^m-q)\cdots(q^m-q^{r-1})]$$

$$\times[(q^{n-r+1}-1)(q^{n-r+2}-1)\cdots(q^n-1)]\}$$

$$\times\{(q^m)^n[(q-1)(q^2-1)\cdots(q^r-1)]\}^{-1}$$

$$=\frac{1}{[m]![n]!}\qquad\text{(by identity (4))}.$$

The proof is completed by noting that the above identity is symmetric in m and n. □

Equipped with the above identity, we are now able to achieve the desired result:

$$\exp_q y\exp_q x=\exp_q[x+y+(1-q^{-1})yx]\qquad\text{if } xy=q^{-1}yx. \tag{6}$$

Proof.

$$\begin{aligned}
\exp_q y \exp_q x &= \left(\sum_{m=0}^{\infty} \frac{y^m}{[m]!}\right)\left(\sum_{n=0}^{\infty} \frac{x^n}{[n]!}\right) \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} y^m x^n \sum_{r=0}^{\min\{m,n\}} \frac{q^{-r(r-1)/2-mn}(q-1)^r}{[m-r]![n-r]![r]!} \qquad \text{(by identity (5))} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\min\{m,n\}} \frac{q^{-r(n-r)} y^r x^{n-r}}{[n-r]!} \cdot \frac{q^{-r(r-1)/2}(1-q^{-1})^r x^r}{[r]!} \cdot \frac{y^{m-r}}{[m-r]!} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\min\{m,n\}} \frac{x^{n-r}}{[n-r]!} \cdot \frac{q^{-r(r-1)/2}(1-q^{-1})^r y^r x^r}{[r]!} \cdot \frac{y^{m-r}}{[m-r]!} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\min\{m,n\}} \frac{x^{n-r}}{[n-r]!} \cdot \frac{(1-q^{-1})^r (yx)^r}{[r]!} \cdot \frac{y^{m-r}}{[m-r]!} \\
&= \left(\sum_{l=0}^{\infty} \frac{x^l}{[l]!}\right)\left(\sum_{k=0}^{\infty} \frac{[(1-q^{-1})yx]^k}{[k]!}\right)\left(\sum_{h=0}^{\infty} \frac{y^h}{[h]!}\right) \\
&= \exp_q x \cdot \exp_q[(1-q^{-1})yx] \cdot \exp_q y \\
&= \exp_q[x + (1-q^{-1})yx] \cdot \exp_q y \\
&\quad \text{(by (3), as } x(1-q^{-1})yx = q^{-1}(1-q^{-1})yxx) \\
&= \exp_q[x + (1-q^{-1})yx + y] \\
&\quad \text{(by (3), as } [x + (1-q^{-1})yx]y = q^{-1}y[x + (1-q^{-1})yx]) \\
&= \exp_q[x + y + (1-q^{-1})yx]. \qquad \square
\end{aligned}$$

Acknowledgments

MYW is grateful to members in the Department of Mathematical Sciences of the University of Durham for all relevant help. This work was partially supported by a British ORS Award as well as a Durham University Research Award.

Appendix

The following is the so-called q-binomial expansion formula and its proof:

$$(x+y)^n = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} x^r y^{n-r} \tag{A1}$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} \equiv \frac{[n]!}{[r]![n-r]!} \qquad [n] \equiv \sum_{k=0}^{n-1} q^k \qquad [0]! \equiv 1$$

subject to the condition that $xy = q^{-1}yx$, q being some complex number.

Proof. Suppose for some $n = k$, we have

$$(x+y)^k = \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix} x^r y^{k-r}.$$

Now consider $(x+y)^{k+1}$,

$$
\begin{aligned}
(x+y)^{k+1} &= \sum_{r=0}^{k} \frac{[k]!}{[r]![k-r]!}(x+y)x^r y^{k-r} \\
&= \sum_{r=0}^{k} \frac{[k]!}{[r]![k-r]!} x^{r+1} y^{k-r} + \sum_{r=0}^{k} \frac{[k]!}{[r]![k-r]!} q^r x^r y^{k-r+1} \\
&= x^{k+1} + \sum_{r=0}^{k-1} \frac{[k]!}{[r]![k-r]!} x^{r+1} y^{k-r} + \sum_{r=0}^{k} \frac{[k]!}{[r]![k-r]!} q^r x^r y^{k-r+1} \\
&= x^{k+1} + \sum_{r=1}^{k} \frac{[k]!}{[r-1]![k-r+1]!} x^r y^{k-r+1} + \sum_{r=0}^{k} \frac{[k]!}{[r]![k-r]!} q^r x^r y^{k-r+1} \\
&= x^{k+1} + \sum_{r=1}^{k} \frac{[k]!(1+q+\cdots+q^{r-1})}{[r]![k-r]!(1+q+\cdots+q^{k-r})} x^r y^{k-r+1} \\
&\quad + \sum_{r=0}^{k} \frac{[k]!}{[r]![k-r]!} q^r x^r y^{k-r+1} \\
&= x^{k+1} + \sum_{r=0}^{k} \frac{[k]!(1+q+\cdots+q^{k})}{[r]![k-r]!(1+q+\cdots+q^{k-r})} x^r y^{k-r+1} \\
&= \sum_{r=0}^{k+1} \frac{[k+1]!}{[r]![k+1-r]!} x^r y^{k+1-r}
\end{aligned}
$$

so the same formula holds for $n = k+1$.

Since for $n = 1$, obviously we have

$$
x + y = \sum_{r=0}^{1} \frac{[1]!}{[r]![1-r]!} x^r y^{1-r}
$$

the proof is complete. □

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